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Many-spinon states and representations of Yangians in the $SU(n)$ Haldane–Shastry model

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Abstract

We study the relation of Yangians and spinons in the $SU(n)$ Haldane–Shastry model. The representation theory of the Yangian is shown to be intimately related to the fractional statistics of the spinons. We construct the spinon Hilbert space from tensor products of the fundamental representations of the Yangian.

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1. Introduction

Quantum groups [1, 2] first arose from the quantum inverse scattering method [3, 4], which had been developed to construct and solve integrable quantum systems. In particular, quantum groups provide a way to construct and study the solutions, called R -matrices, of the quantum Yang–Baxter equation. Mathematically, quantum groups are deformations of the universal enveloping algebra of the classical Lie algebras. In general, they depend on a parameter h and the underlying Lie algebra is recovered in the limit $h \rightarrow 0$. Yangians are special quantum groups which were first introduced by Drinfel'd in 1985 [1]. Their representation theory is intimately related [5, 6] to the rational R -matrices.

Later it was discovered that Yangians also appear as additional symmetries of quantum field theories [7, 8], and furthermore, that Yangians are part of the symmetry algebra of special integrable spin systems. In particular, the one-dimensional nearest-neighbour Heisenberg model possesses a Yangian symmetry in the limit of a chain of infinite length [9], whereas the Haldane–Shastry model possesses a Yangian symmetry even for a chain of finite length [8, 10]. In addition, a Yangian symmetry exists for the one-dimensional Hubbard model on an infinite chain [11] as well as for a finite chain with suitable hopping amplitudes [12, 13].

From a physical point of view, the Haldane–Shastry model (HSM) [14] owes its special importance to two reasons. The first and more technical one is that the model is exactly solvable even for a chain of finite length. It is possible to derive explicit wavefunctions for the ground

state and the elementary spinon excitations [15]. The second and more fundamental reason is that the HSM possesses non-interacting or free spinon excitations [16], a conclusion which is in particular supported by the trivial spinon–spinon scattering matrix calculated by Essler [17] using the asymptotic Bethe Ansatz. In 2001, this picture was challenged by Bernevig *et al* [18], who studied the explicit two-spinon wavefunctions and claimed to have identified effects of an interaction between the spinons. A critical re-examination [19] of their conclusions, however, showed that these alleged interaction effects are in fact due to the fractional statistics of the spinons [20], which results in non-trivial quantization rules for the individual spinon momenta [21]. This debate showed that free particles may appear interacting at first sight if an inappropriate representation is chosen.

In this paper, we investigate the relation between the Yangian symmetry and the physical properties of the spinons. We show that individual spinons in the HSM transform under the fundamental representation of the Yangian. We then study the implications of the Yangian symmetry on many-spinon states. The main result of this analysis is the derivation of a general rule governing the fractional statistics of the spinons. This rule states that in the spinon Hilbert space only the irreducible subrepresentations of the tensor products of certain fundamental representations of the Yangian exist. This enables us to derive, starting from a set of individual spinon momenta, the allowed values of the total spin of the corresponding many-spinon states, which are subject to highly non-trivial restrictions due to the fractional statistics of the spinons. All results are generalized to the elementary excitations of the SU(3) HSM.

Before we discuss the main topic of this paper, we will briefly review the HSM and its most important physical features, and give a concise introduction to Yangians and their representation theory.

2. Haldane–Shastry model

In 1988, Haldane and Shastry discovered independently [14] that a trial wavefunction proposed by Gutzwiller [22] in 1963 provides the exact ground state to a Heisenberg-type spin Hamiltonian whose interaction strength falls off as the inverse square of the distance between two spins on the chain. Later the model was generalized to an SU(n) spin by Kawakami [23].

The HSM is most conveniently formulated by embedding the one-dimensional chain with periodic boundary conditions into the complex plane by mapping it onto the unit circle with the (SU(n)) spins located at complex positions $\eta_\alpha = \exp(i\frac{2\pi}{N}\alpha)$, where N denotes the number of sites and $\alpha = 1, \dots, N$. The Hamiltonian is given by [14]

$$H_{\text{HS}} = \frac{2\pi^2}{N^2} \sum_{\alpha \neq \beta}^N \frac{\mathbf{S}_\alpha \cdot \mathbf{S}_\beta}{|\eta_\alpha - \eta_\beta|^2}. \quad (1)$$

The SU(3) HSM [23] is given by replacing \mathbf{S}_α by the eight-dimensional SU(3) spin vector $\mathbf{J}_\alpha = \frac{1}{2} \sum_{\sigma\tau} c_{\alpha\sigma}^\dagger \boldsymbol{\lambda}_{\sigma\tau} c_{\alpha\tau}$, where $\boldsymbol{\lambda}$ is a vector consisting of the eight Gell–Mann matrices [24], and σ and τ are SU(3) spin or colour indices which take the values blue (b), red (r) or green (g) (see figure 1(a)). The spins on the lattice sites transform under the fundamental representation n of SU(n), e.g. $S = 1/2$ for SU(2).

The ground state ($N = 2M$, M integer) of the SU(2) model is given by

$$|\Psi_0\rangle = P_G |\Psi_{\text{SD}}^N\rangle, \quad |\Psi_{\text{SD}}^N\rangle \equiv \prod_{q \in \mathcal{I}} c_{q\uparrow}^\dagger c_{q\downarrow}^\dagger |0\rangle, \quad (2)$$

where the Gutzwiller projector P_G eliminates configurations with more than one particle on any site and the interval \mathcal{I} contains M adjacent momenta. For SU(n), each momentum in \mathcal{I} has to be occupied by n particles with different spins [25].

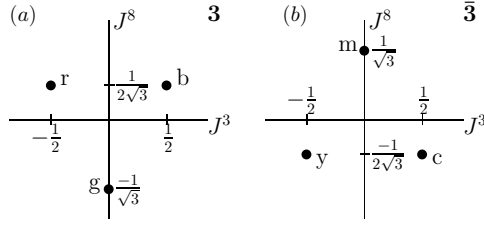


Figure 1. Weight diagrams of the three-dimensional representations of SU(3). J^3 and J^8 span the Cartan subalgebra of $\mathfrak{su}(3)$ [24].

The model is invariant under global SU(2) or SU(3) rotations generated by

$$\mathbf{S} = \sum_{\alpha=1}^N \mathbf{S}_{\alpha} \quad (\text{for SU(2)}) \quad \text{or} \quad \mathbf{J} = \sum_{\alpha=1}^N \mathbf{J}_{\alpha} \quad (\text{for SU(3)}), \quad (3)$$

respectively. The system possesses an additional symmetry [8, 10], which is given by

$$\Lambda = \frac{i}{2} \sum_{\alpha \neq \beta}^N \frac{\eta_{\alpha} + \eta_{\beta}}{\eta_{\alpha} - \eta_{\beta}} (\mathbf{S}_{\alpha} \times \mathbf{S}_{\beta}) \quad \text{or} \quad \Lambda^a = \frac{1}{2} \sum_{\alpha \neq \beta}^N \frac{\eta_{\alpha} + \eta_{\beta}}{\eta_{\alpha} - \eta_{\beta}} f^{abc} J_{\alpha}^b J_{\beta}^c \quad (4)$$

(we use the Einstein summation convention) where $a, b, c = 1, \dots, 8$ and f^{abc} denote the structure constants of SU(3). The total spin (3) and rapidity operator (4) generate the Yangian, which we will discuss in detail below.

The elementary excitations of the SU(n) HSM are constructed by annihilation of a particle from the Slater determinant state before Gutzwiller projection [15, 26],

$$|\Psi_{p\bar{\sigma}}\rangle = P_G c_{-p\sigma} |\Psi_{SD}^{N+1}\rangle, \quad N = nM - 1. \quad (5)$$

Here p denotes the momentum, σ either the spin (for $n = 2$) or one of the colors blue, red or green (for $n = 3$). In order to ensure that every site is occupied by a spin after the projection, we annihilate a particle from the Slater determinant state with $N + 1$ particles. Note that for SU(2) the annihilation of an up-spin electron creates a down-spin spinon and vice versa. The spinons possess spin 1/2 like the electrons on the lattice sites. For SU(3) the situation is, however, fundamentally different. Here, the annihilation of a , say, blue particle creates an anti-blue SU(3) spinon or coloron. (We will use the terms SU(3) spinon and coloron simultaneously.) This means that colorons transform under the conjugate representation $\bar{\mathbf{3}}$ (see figure 1(b)), if the particles on the sites transform under $\mathbf{3}$ [26, 27].

A non-orthogonal but complete basis for spin-polarized two-spinon eigenstates with total momentum $p = -k_1 - k_2$ is given by (we assume $k_1 > k_2$)

$$|\Psi_{p_1\bar{\sigma}, p_2\bar{\sigma}}\rangle = P_G c_{k_1\sigma} c_{k_2\sigma} |\Psi_{SD}^{N+2}\rangle, \quad N = nM - 2. \quad (6)$$

These states are not energy eigenstates, but as H_{HS} scatters k_1 and k_2 in only one direction (increasing $k_1 - k_2$), there is a one-to-one correspondence between these basis states and the exact eigenstates of the HSM. The total energy of the eigenstates takes the form of free particles if and only if the single-spinon momenta are shifted with respect to $k_{1,2}$ [19, 26]:

$$p_{1,2} = -k_{1,2} \pm \frac{1}{2n} \frac{2\pi}{N}, \quad p_1 < p_2. \quad (7)$$

The shift is due to the fractional statistics of the spinons [20, 21]. In general, all two-spinon states with the same single-spinon momenta are obtained by acting with the total spin and

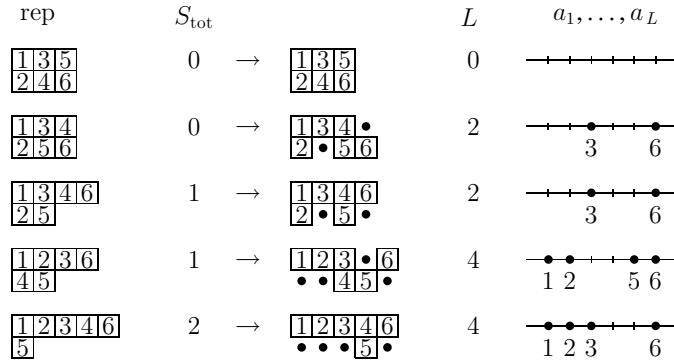


Figure 2. Examples of eigenstates of the SU(2) HSM with $N = 6$ sites in terms of spinons. The dots represent the spinons, the first tableau is the ground state.

rapidity operators on the polarized states (6). In particular, for SU(2) the action of $\Lambda^z S^-$ yields the two-spinon singlet states. However, this two-spinon singlet state $\Lambda^z S^- |\Psi_{p_1 \uparrow, p_2 \uparrow}\rangle$ exists only for $p_2 - p_1 > \frac{1}{2} \frac{2\pi}{N}$, as (6) is annihilated by $\Lambda^z S^-$ for $p_2 - p_1 = \frac{1}{2} \frac{2\pi}{N}$ [8]. For general many-spinon states or spinons in the SU(n) chain these restrictions on the possible values of the total spin become highly non-trivial.

3. Tableau formalism

Recently, a formalism was introduced to obtain all existing many-spinon states starting from a given set of single-spinon momenta [28]. The formalism works as follows. To begin with, the Hilbert space of a system of N identical SU(n) spins can be decomposed into representations of the total spin (3), which commutes with (1) and hence can be used to classify the energy eigenstates. These representations are compatible with the representations of the symmetric group S_N of N elements, which may be expressed in terms of Young tableaux [29]. In order to obtain these Young tableaux, we draw for each of the N spins a box numbered consecutively from left to right. The representations of SU(n) are constructed by putting the boxes together such that the numbers assigned to them increase in each row from left to right and in each column from top to bottom. Each tableau obtained in this way represents an irreducible representation of SU(n); it further indicates symmetrization over all boxes in the same row, and antisymmetrization over all boxes in the same column. This implies that we cannot have more than n boxes on top of each other for SU(n) spins.

Now, there is a one-to-one correspondence between these Young tableaux and the non-interacting many-spinon states, i.e., the eigenstates of the HSM. The principle is illustrated for a few representations of an SU(2) chain with six sites in figure 2, and for an SU(3) chain in figure 3. In each Young tableau we shift boxes to the right such that each box is below or in the column to the right of the box with the preceding number. Each missing box in the resulting, extended tableaux represents a spinon, to which we assign a spinon momentum number (SMN) a_i as follows: for an SU(2) chain, it is simply given by the number in the box in the same column. For a general SU(n) chain, the SMN's for the spinons in each column are given by a sequence of numbers (half-integers for n odd, integers for n even) with integer spacings such that the arithmetic mean equals the arithmetic mean of the numbers in the boxes of the column, as illustrated in the examples presented in figure 3. The extended tableaux

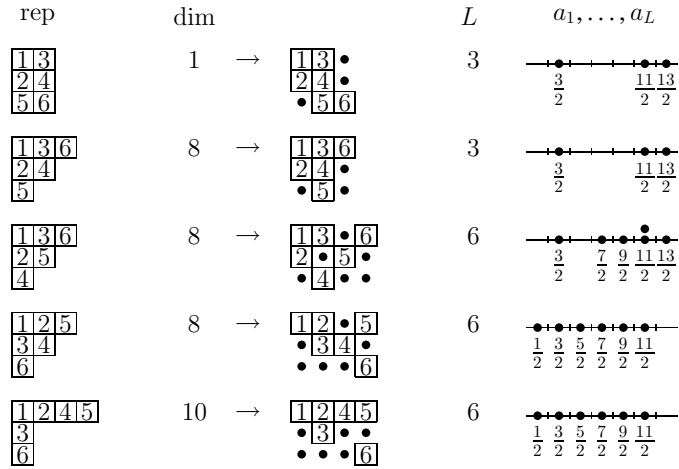


Figure 3. Examples of eigenstates of the SU(3) HSM with $N = 6$ sites in terms of colorons.

provide us with the total SU(n) spin of each multiplet (given by the original Young tableau), as well as the number L of spinons present and the individual single-spin momenta p_1, \dots, p_L given in terms of the SMN's as

$$p_i = \frac{2\pi}{N} \frac{a_i - \frac{1}{2}}{n}, \tag{8}$$

which implies $0 \leq p_i \leq \frac{2\pi}{n}$ for $N \rightarrow \infty$. The total momentum and Haldane–Shastry energies of the many-spinon states are

$$p = p_0 + \sum_{i=1}^L p_i, \quad E = E_0 + \sum_{i=1}^L \epsilon(p_i), \tag{9}$$

where p_0 and E_0 denote the ground-state momentum and energy given by

$$p_0 = -\frac{(n-1)\pi}{n} N, \quad E_0 = -\frac{\pi^2}{12} \left(\frac{n-2}{n} N + \frac{2n-1}{N} \right), \tag{10}$$

and the single-spinon dispersion is

$$\epsilon(p) = \frac{n}{4} p \left(\frac{2\pi}{n} - p \right) + \frac{n^2 - 1}{12n} \frac{\pi^2}{N^2}. \tag{11}$$

This formalism provides the complete spectrum of the HSM [28]. It is easy to see that the momentum spacings for spin-polarized spinons predicted by this formalism reproduce (7) for general n , and that spinons transform under the representation \bar{n} of SU(n).

4. Yangians

In this section, we discuss the Yangian of \mathfrak{sl}_n . Let $\{I^a\}$ be an orthonormal basis with respect to a scalar product $\langle \cdot, \cdot \rangle$ of \mathfrak{sl}_n . We will use

$$\langle X, Y \rangle = \text{tr}(XY), \quad X, Y \in \mathfrak{sl}_n. \tag{12}$$

For example, an orthonormal basis of \mathfrak{sl}_2 is then given by $I^a = \sqrt{2}S^a = \sigma^a/\sqrt{2}$ with the Pauli matrices σ^a , $a = 1, 2, 3$. The operators I^a fulfil the algebra

$$[I^a, I^b] = c^{abc} I^c, \quad a, b, c = 1, \dots, n^2 - 1, \quad (13)$$

where c^{abc} are the structure constants, e.g. $c^{abc} = i\sqrt{2}\varepsilon^{abc}$ for \mathfrak{sl}_2 .

The Yangian $Y(\mathfrak{sl}_n)$ [1] of \mathfrak{sl}_n is the infinite-dimensional associative algebra over \mathbb{C} generated by the elements I^a, \mathcal{T}^a with defining relations

$$[I^a, I^b] = c^{abc} I^c, \quad [I^a, \mathcal{T}^b] = c^{abc} \mathcal{T}^c, \quad (14)$$

$$[\mathcal{T}^a, [\mathcal{T}^b, I^c]] - [I^a, [\mathcal{T}^b, \mathcal{T}^c]] = c^{abcdef} \{I^d, I^e, I^f\}, \quad (15)$$

$$[[\mathcal{T}^a, \mathcal{T}^b], [I^i, \mathcal{T}^j]] + [[\mathcal{T}^i, \mathcal{T}^j], [I^a, \mathcal{T}^b]] = (c^{abcdef} c^{ijc} + c^{ijdef} c^{abc}) \{I^d, I^e, \mathcal{T}^f\}, \quad (16)$$

where repeated indices are summed over and

$$c^{abcdef} = \frac{1}{24} c^{adi} c^{bej} c^{cfk} c^{ijk}, \quad \{X^a, X^b, X^c\} = \sum_{\pi \in \mathfrak{S}_3} X^{\pi(a)} X^{\pi(b)} X^{\pi(c)}. \quad (17)$$

$Y(\mathfrak{sl}_n)$ has a Hopf algebra structure with comultiplication $\Delta : Y(\mathfrak{sl}_n) \rightarrow Y(\mathfrak{sl}_n) \otimes Y(\mathfrak{sl}_n)$ given by

$$\Delta(I^a) = I^a \otimes 1 + 1 \otimes I^a, \quad \Delta(\mathcal{T}^a) = \mathcal{T}^a \otimes 1 + 1 \otimes \mathcal{T}^a - \frac{1}{2} c^{abc} I^b \otimes I^c. \quad (18)$$

The counit $\epsilon : Y(\mathfrak{sl}_n) \rightarrow \mathbb{C}$ and the antipode $\mathcal{S} : Y(\mathfrak{sl}_n) \rightarrow Y(\mathfrak{sl}_n)$ will not be used in this paper; there definitions can be found in the literature [1, 6]. The defining relations (14)–(16) depend on the choice of the scalar product (12) but, up to isomorphism, the Hopf algebra $Y(\mathfrak{sl}_n)$ does not [6]. We have chosen (12) in order to match the notations of [5, 30] for the representation theory of $Y(\mathfrak{sl}_n)$.

The algebra generated by the total spin (3) and rapidity operator (4) is recovered with the identifications

$$S^a = \frac{1}{\sqrt{2}} I^a \quad \text{or} \quad J^a = \frac{1}{\sqrt{2}} I^a, \quad \Lambda^a = \frac{1}{\sqrt{2}} \mathcal{T}^a, \quad (19)$$

hence, the Yangian constitutes a symmetry of the HSM [8, 10]. The comultiplication defines the action of the operators (19) on two-particle states, and being a homomorphism it is consistent on three-particle states.

There is another realization of the Yangian, first given by Drinfel'd in 1988 [31], which will be used to set the representation theory of the Yangian in the following section. It is based on the Cartan–Weyl basis [24] of \mathfrak{sl}_n , which for $n = 2$ is given in terms of the spin operators by

$$H_1 = 2S^z, \quad X_1^\pm = S^\pm = S^x \pm iS^y, \quad (20)$$

whereas for $n = 3$ we have explicitly

$$H_1 = 2J^3, \quad H_2 = \sqrt{3}J^8 - J^3, \quad (21)$$

$$X_1^\pm = I^\pm = J^1 \pm iJ^2, \quad X_2^\pm = U^\pm = J^6 \pm iJ^7, \quad V^\pm = J^4 \pm iJ^5. \quad (22)$$

The operators I^\pm and U^\pm are sufficient to span \mathfrak{sl}_3 as we can re-express the non-simple root as $V^\pm = I^\pm + U^\pm$.

With the identifications $H_{i,0} = H_i$ and $X_{i,0}^\pm = X_i^\pm$ the relations of the Yangian $Y(\mathfrak{sl}_n)$ read as follows [6, 31] ($1 \leq i \leq n - 1, k \in \mathbb{N}_0$):

$$[H_{i,k}, H_{j,l}] = 0, \quad [H_{i,0}, X_{j,k}^\pm] = \pm B_{ij} X_{j,k}^\pm, \quad [X_{i,k}^+, X_{j,l}^-] = \delta_{ij} H_{i,k+l}, \quad (23)$$

$$[H_{i,k+1}, X_{j,l}^\pm] - [H_{i,k}, X_{j,l+1}^\pm] = \pm \frac{1}{2} B_{ij} (H_{i,k} X_{j,l}^\pm + X_{j,l}^\pm H_{i,k}), \tag{24}$$

$$[X_{i,k+1}^\pm, X_{j,l}^\pm] - [X_{i,k}^\pm, X_{j,l+1}^\pm] = \pm \frac{1}{2} B_{ij} (X_{i,k}^\pm X_{j,l}^\pm + X_{j,l}^\pm X_{i,k}^\pm), \tag{25}$$

$$[X_{i,k_1}^\pm, [X_{i,k_2}^\pm, X_{j,l}^\pm]] + [X_{i,k_2}^\pm, [X_{i,k_1}^\pm, X_{j,l}^\pm]] = 0, \quad \text{for } i = \pm(j+1), \tag{26}$$

$$[X_{i,k}^\pm, X_{j,l}^\pm] = 0, \quad \text{for } i \neq j, \pm(j+1), \tag{27}$$

where $B_{ii} = 2$, $B_{i,i+1} = B_{i,i-1} = -1$ and $B_{ij} = 0$ otherwise.

For simplicity we state the isomorphism between the two realizations of $Y(\mathfrak{sl}_n)$ only for the diagonal operators in the cases $n = 2$ and $n = 3$. For $Y(\mathfrak{sl}_2)$ it is given by

$$S^z \mapsto \frac{1}{2} H_{1,0}, \quad \Lambda^z \mapsto \frac{1}{2} H_{1,1} + \frac{1}{4} (S^+ S^- + S^- S^+ - H_{1,0}^2), \tag{28}$$

while for $Y(\mathfrak{sl}_3)$ it reads

$$J^3 \mapsto \frac{1}{2} H_{1,0}, \quad J^8 \mapsto \frac{1}{2\sqrt{3}} H_{1,0} + \frac{1}{\sqrt{3}} H_{2,0}, \tag{29}$$

$$\Lambda^3 \mapsto \frac{1}{2} H_{1,1} - \frac{1}{4} H_{1,0}^2 + \frac{1}{4} (I^+ I^- + I^- I^+) - \frac{1}{8} (U^+ U^- + U^- U^+ - V^+ V^- - V^- V^+), \tag{30}$$

$$\begin{aligned} \Lambda^8 \mapsto & \frac{1}{2\sqrt{3}} H_{1,1} + \frac{1}{\sqrt{3}} H_{2,1} - \frac{1}{4\sqrt{3}} H_{1,0}^2 - \frac{1}{2\sqrt{3}} H_{2,0}^2 \\ & + \frac{\sqrt{3}}{8} (U^+ U^- + U^- U^+ + V^+ V^- + V^- V^+). \end{aligned} \tag{31}$$

Here, we have used the short-hand notations $S^\pm = X_{1,0}^\pm$ as well as $I^\pm = X_{1,0}^\pm$, $U^\pm = X_{2,0}^\pm$ and $V^\pm = X_{1,0}^\pm + X_{2,0}^\pm$.

5. Representations of Yangians

The representation theory of $Y(\mathfrak{sl}_n)$ [5, 6, 30, 32] is based on the existence of the evaluation homomorphism, which connects $Y(\mathfrak{sl}_n)$ with the universal enveloping algebra $U(\mathfrak{sl}_n)$ of \mathfrak{sl}_n . For all $\xi \in \mathbb{C}$, $\text{ev}_\xi : Y(\mathfrak{sl}_n) \rightarrow U(\mathfrak{sl}_n)$ is given by

$$I^a \mapsto I^a, \quad \mathcal{I}^a \mapsto \xi I^a + \frac{1}{4} \sum_{b,c=1}^{n^2-1} \text{tr}(I^a (I^b I^c + I^c I^b)) I^b I^c, \tag{32}$$

where the trace is computed by representing I^a as $n \times n$ matrices. In general, given a representation of \mathfrak{sl}_n one obtains a one-parameter family of $Y(\mathfrak{sl}_n)$ representations via the pull-back of the evaluation homomorphism. Explicitly, if ϕ is a representation of \mathfrak{sl}_n on V , $\phi_\xi = \phi \circ \text{ev}_\xi$ is a representation of $Y(\mathfrak{sl}_n)$ on V , which is called the evaluation representation with the spectral parameter ξ .

A representation V of the Yangian $Y(\mathfrak{sl}_n)$ is said to be highest weight, if there exists a vector $v \in V$ such that $V = Y(\mathfrak{sl}_n)v$ and

$$X_{i,k}^+ v = 0, \quad H_{i,k} v = d_{i,k} v, \tag{33}$$

with an array of complex numbers $(d_{i,k})$. In this case, v is called the Yangian highest weight state (YHWS) of V and $(d_{i,k})$ its highest weight. As it was shown by Drinfel'd [31], the irreducible highest weight representation V of $Y(\mathfrak{sl}_n)$ with highest weight $(d_{i,k})$ is finite dimensional if and only if there exist monic polynomials $P_i \in \mathbb{C}[u]$, $1 \leq i \leq n-1$ such that

$$\frac{P_i(u+1)}{P_i(u)} = 1 + \sum_{k=0}^{\infty} \frac{d_{i,k}}{u^{k+1}}, \tag{34}$$

in the sense that the right-hand side is a Laurent expansion of the left-hand side about $u = \infty$ [30]. Hence, the Drinfel'd polynomials $P_i(u)$ classify the finite-dimensional irreducible representations of the Yangian. The i th fundamental representation of $Y(\mathfrak{sl}_n)$ with spectral parameter $\xi \in \mathbb{C}$ is defined as the irreducible highest weight representation with Drinfel'd polynomials

$$P_i(u) = u - \xi, \quad P_j(u) = 1 \quad \text{for } j \neq i. \quad (35)$$

We will denote the fundamental representation of $Y(\mathfrak{sl}_2)$ with Drinfel'd polynomial $P(u) = u - \xi$ by $V(\frac{1}{2}, \xi)$. It can be constructed explicitly as the pull-back of the \mathfrak{sl}_2 representation $\frac{1}{2}$ under ev_ξ . For $Y(\mathfrak{sl}_3)$ we denote by $V(\mathbf{3}, \xi)$ and $V(\bar{\mathbf{3}}, \xi)$ the three-dimensional representations with Drinfel'd polynomials $P_1(u) = u - \xi, P_2(u) = 1$ and $P_1(u) = 1, P_2(u) = u - \xi$, respectively. If $V(\mathbf{3}, \xi)$ and $V(\bar{\mathbf{3}}, \xi)$ are realized as evaluation representations, we obtain an additional shift in the spectral parameter due to the trace on the right-hand side of (32). For example, $V(\bar{\mathbf{3}}, \xi)$ is obtained as an evaluation representation of the \mathfrak{sl}_3 representation $\bar{\mathbf{3}}$ with evaluation parameter $\xi + 2/3$ (see appendix A).

5.1. Representation theory of $Y(\mathfrak{sl}_2)$

In the following, we denote the evaluation representation of the $(m + 1)$ -dimensional \mathfrak{sl}_2 representation $\frac{m}{2}$ with spectral parameter $\xi \in \mathbb{C}$ by $V(\frac{m}{2}, \xi)$. The Drinfel'd polynomial of $V(\frac{m}{2}, \xi)$ is given by [5]

$$P(u) = \left(u - \xi + \frac{m-1}{2}\right) \left(u - \xi + \frac{m-3}{2}\right) \cdots \left(u - \xi - \frac{m-1}{2}\right). \quad (36)$$

Now, let V_1 and V_2 be two representations of $Y(\mathfrak{sl}_2)$. The action of $X \in Y(\mathfrak{sl}_2)$ on the tensor product $V_1 \otimes V_2$ is given by $\Delta(X)$, where Δ is the comultiplication (18). We stress that due to the last term of $\Delta(\mathcal{I}^a)V_1$ and V_2 are intertwined. In particular, as Δ is not commutative, $V_1 \otimes V_2$ and $V_2 \otimes V_1$ are not isomorphic in general. In all cases, however, if v_1^+ and v_2^+ are the YHWS's of V_1 and V_2 , respectively, the vector $v_1^+ \otimes v_2^+$ will be the YHWS of $V_1 \otimes V_2$. The action of $Y(\mathfrak{sl}_2)$ on r -fold tensor products is defined by repeated application of Δ .

The central theorem on the tensor product $V = V(\frac{m_1}{2}, \xi_1) \otimes V(\frac{m_2}{2}, \xi_2)$ is due to Chari and Pressley [5]:

- (i) If $|\xi_1 - \xi_2| \neq \frac{m_1+m_2}{2} - p + 1$ for all $p \in \mathbb{N}$ with $0 < p \leq \min(m_1, m_2)$, then V is irreducible as $Y(\mathfrak{sl}_2)$ representation.
- (ii) If $\xi_2 - \xi_1 = \frac{m_1+m_2}{2} - p + 1$ for some $p \in \mathbb{N}$ with $0 < p \leq \min(m_1, m_2)$, then V has a unique proper $Y(\mathfrak{sl}_2)$ subrepresentation W generated by the highest weight vector of V . Explicitly, we have

$$W \cong V\left(\frac{p-1}{2}, \xi_1 + \frac{m_1-p+1}{2}\right) \otimes V\left(\frac{m_1+m_2-p+1}{2}, \xi_2 - \frac{m_1-p+1}{2}\right)$$

$$\text{and, as representation of } \mathfrak{sl}_2, W \cong \frac{m_1+m_2}{2} \oplus \cdots \oplus \frac{m_1+m_2-2p+2}{2}.$$

The third case, $\xi_1 - \xi_2 = \frac{m_1+m_2}{2} - p + 1$ for some $p \in \mathbb{N}$ with $0 < p \leq \min(m_1, m_2)$, was also discussed in [5], but will not be used in the study of the HSM.

In order to illustrate the two different situations, we consider the simplest non-trivial case $V = V(\frac{1}{2}, \xi_1) \otimes V(\frac{1}{2}, \xi_2)$ for $\xi_1 < \xi_2$. Clearly, regarded as representation of \mathfrak{sl}_2 V it decomposes as $V \cong \mathbf{1} \oplus \mathbf{0}$. If $\xi_2 - \xi_1 \neq 1$, then V is irreducible as $Y(\mathfrak{sl}_2)$ representation. One can always find an operator in $Y(\mathfrak{sl}_2)$ which yields a singlet state when acting on a triplet state and vice versa. If $\xi_2 - \xi_1 = 1$, however, V contains a proper $Y(\mathfrak{sl}_2)$ subrepresentation $W \cong \mathbf{1}$ generated by the YHWS. In particular, there exists no operator in $Y(\mathfrak{sl}_2)$ which yields

the singlet state when acting on a triplet state. However, it is possible to obtain a triplet state when acting on the spin singlet state with an appropriate operator. Hence, V is not a direct sum of irreducible $Y(\mathfrak{sl}_2)$ representations.

The highest weight of $V = V\left(\frac{m_1}{2}, \xi_1\right) \otimes V\left(\frac{m_2}{2}, \xi_2\right)$ is obtained from its Drinfel'd polynomial, which is in the irreducible case simply given by the product of the original polynomials [5]. In the reducible case the highest weight is determined by the highest weight component of V using (36).

5.2. Representation theory of $Y(\mathfrak{sl}_3)$

The representation theory of $Y(\mathfrak{sl}_3)$ is not known in the same detail as it is for $Y(\mathfrak{sl}_2)$. Although there exist irreducibility criteria for tensor products of evaluation representations of $Y(\mathfrak{sl}_n)$ [32], an explicit description of the irreducible subrepresentation of such tensor products including its spectral parameter is missing. We will restrict ourselves here to tensor products of two fundamental representations of $Y(\mathfrak{sl}_3)$. There are three different situations [30]:

- (i) $V = V(\mathbf{3}, \xi_1) \otimes V(\mathbf{3}, \xi_2)$ is reducible as $Y(\mathfrak{sl}_3)$ representation if and only if $\xi_1 - \xi_2 = \pm 1$. If $\xi_2 - \xi_1 = 1$, then V has a proper $Y(\mathfrak{sl}_3)$ subrepresentation W generated by the highest weight vector of V and, as representation of \mathfrak{sl}_3 , $W \cong \mathbf{6}$.
- (ii) $V = V(\mathbf{3}, \xi_1) \otimes V(\bar{\mathbf{3}}, \xi_2)$ (or $V = V(\bar{\mathbf{3}}, \xi_1) \otimes V(\mathbf{3}, \xi_2)$) is reducible as $Y(\mathfrak{sl}_3)$ representation if and only if $\xi_1 - \xi_2 = \pm 3/2$. If $\xi_2 - \xi_1 = 3/2$, then V has a proper $Y(\mathfrak{sl}_3)$ subrepresentation W generated by the highest weight vector of V and, as representation of \mathfrak{sl}_3 , $W \cong \mathbf{8}$.
- (iii) $V = V(\bar{\mathbf{3}}, \xi_1) \otimes V(\bar{\mathbf{3}}, \xi_2)$ is reducible as $Y(\mathfrak{sl}_3)$ representation if and only if $\xi_1 - \xi_2 = \pm 1$. If $\xi_1 - \xi_2 = 1$, then V has a proper $Y(\mathfrak{sl}_3)$ subrepresentation W not containing the highest weight vector of V and, as representation of \mathfrak{sl}_3 , $W \cong \mathbf{3}$. If $\xi_2 - \xi_1 = 1$, then V has a proper $Y(\mathfrak{sl}_3)$ subrepresentation W generated by the highest weight vector of V and, as representation of \mathfrak{sl}_3 , $W \cong \bar{\mathbf{6}}$.

If the tensor product $V = V_1 \otimes V_2$ is irreducible, the Drinfel'd polynomials of V are given by the products of the polynomials of V_1 and V_2 [6]. Furthermore, we show in appendix B that the proper $Y(\mathfrak{sl}_3)$ subrepresentation W of $V(\bar{\mathbf{3}}, \xi) \otimes V(\bar{\mathbf{3}}, \xi - 1)$ is given by $W \cong V(\mathbf{3}, \xi - \frac{1}{2})$.

6. Spinons and representations of $Y(\mathfrak{sl}_2)$

As mentioned above it is well known [8, 10] that the $SU(2)$ HSM possesses the Yangian symmetry $Y(\mathfrak{sl}_2)$ and therefore that its Hilbert space decomposes into irreducible representations of $Y(\mathfrak{sl}_2)$. It is also known [28] how to build up the Hilbert space of the HSM by non-interacting spinon excitations. In this section, we will study the relation between these many-spinon states and representations of $Y(\mathfrak{sl}_2)$.

6.1. One-spinon states

We first derive the relation between the one-spinon states and the fundamental representations of $Y(\mathfrak{sl}_2)$. Consider a chain with an odd number of sites N . The individual one-spinon momenta are given by [15, 18]

$$p = \frac{\pi}{2} - \frac{2\pi}{N} \left(\mu + \frac{1}{4} \right), \quad 0 \leq \mu \leq \frac{N-1}{2}, \quad (37)$$

where we have assumed $N - 1$ to be divisible by four, and thereby restricted the momentum to $-\pi/2 \leq p \leq \pi/2$. The up-spin spinons are YHWS's (they are annihilated by S^+ , Λ^+ , ...), their spin currents are given by [18]

$$\Lambda^z |\uparrow\rangle = \left(\frac{N-1}{4} - \mu \right) |\uparrow\rangle. \tag{38}$$

Here $|\uparrow\rangle$ denotes the state with one up-spin spinon.

On the other hand the one-spinon states are represented by tableaux of the form

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline & & \bullet & & & & & \\ \hline & & a & & & & & \\ \hline \end{array} \quad a = N - 2\mu, \quad 0 \leq \mu \leq \frac{N-1}{2},$$

where we omit the superfluous numbers in the boxes of the tableaux. Note that (37) is recovered using (8)–(10). Now, (28) together with (38) yields

$$H_{1,1} |\uparrow\rangle = \left(2\Lambda^z - \frac{1}{2}[S^+S^- + S^-S^+ - 4(S^z)^2] \right) |\uparrow\rangle = \left(a - \frac{N+1}{2} \right) |\uparrow\rangle, \tag{39}$$

where the term in squared brackets vanishes as the spinon possesses spin $S = 1/2$.

Hence, individual spinons transform under the $Y(\mathfrak{sl}_2)$ representation $V(\frac{1}{2}, \xi)$, where the spectral parameter ξ is in terms of the SMN given by

$$\xi = a - \frac{N+1}{2}, \quad -\frac{N-1}{2} \leq \xi \leq \frac{N-1}{2}. \tag{40}$$

6.2. Two-spinon states

Let us consider two spin-polarized spinons represented by the tableau

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline & & \bullet & & \bullet & & & \\ \hline & & a_1 & & & & & \\ \hline \end{array} \quad a_1 = N - 2\mu - 1, \quad a_2 = N - 2\nu, \\ 0 \leq \nu \leq \mu \frac{N-2}{2}$$

Note that, as N is even, a_1 is always odd and a_2 is always even. There are two fundamentally different cases. If $a_2 - a_1 > 1$, there exists a two-spinon singlet with the same SMN's (and hence the same energy), whereas for $a_2 - a_1 = 1$ there exists no corresponding singlet. Graphically we have

$$a_2 - a_1 > 1 : \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline & & \bullet & & & \bullet & & \\ \hline & & & & & & & \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & \bullet & \\ \hline & & \bullet & & & & & \\ \hline & & & & & & & \\ \hline \end{array} \\ a_2 - a_1 = 1 : \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline & & & \bullet & \bullet & & & \\ \hline & & & & & & & \\ \hline \end{array} \quad \text{only.}$$

This can be understood by applying the representation theory of $Y(\mathfrak{sl}_2)$. In both cases the two spinons transform under the product representation $V = V(\frac{1}{2}, \xi_1) \otimes V(\frac{1}{2}, \xi_2)$, where the spectral parameters are given by $\xi_{1,2} = a_{1,2} - (N+1)/2$, respectively. In the first case we have $\xi_2 - \xi_1 > 1$, hence by section 5.1.i V is irreducible as $Y(\mathfrak{sl}_2)$ representation. As \mathfrak{sl}_2 representation we have $V \cong \mathbf{1} \oplus \mathbf{0}$, i.e., the triplet and singlet represented by the two tableaux shown above. V is generated by its YHWS, which is the spin-polarized two-spinon state $|\uparrow\uparrow\rangle = |\uparrow\rangle \otimes |\uparrow\rangle$. In particular, the operator $\Lambda^z S^- \in Y(\mathfrak{sl}_2)$ yields the two-spinon singlet state when acting on the YHWS, $\Lambda^z S^- |\uparrow\uparrow\rangle \propto |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$, while leaving the individual spinon momenta and hence the energy unchanged.

In the second case we have $\xi_2 - \xi_1 = 1$, and by section 5.1.ii V is reducible. The YHWS of V , which is again $|\uparrow\uparrow\rangle$, generates the proper $Y(\mathfrak{sl}_2)$ subrepresentation $W \cong \mathbf{1}$, i.e., only the triplet states are generated by $|\uparrow\uparrow\rangle$ and, in particular, it is $\Lambda^z S^- |\uparrow\uparrow\rangle = 0$. This is reflected by the existence of only one tableau if the SMN's fulfil $a_2 - a_1 = 1$, and is consistent with results drawn from the asymptotic Bethe Ansatz for the HSM [8, 33] as well as conformal field theory spectra [34].

The loss of the singlet, i.e., its non-existence in the spinon Hilbert space, is a manifestation of the fractional statistics of the spinons. The momentum spacings for two spinons with individual momenta p_1 and p_2 ($p_2 > p_1$) are in general given by $p_2 - p_1 = 2\pi(1/2 + \ell)/N$, $\ell \in \mathbb{N}_0$ [19]. When the spinons occupy adjacent momenta, $p_2 - p_1 = \pi/N$, only the triplet exists, which is mathematically implemented by the requirement of irreducibility under $Y(\mathfrak{sl}_2)$ transformations. In analogy, the Pauli principle enforces two electrons with identical momenta to form a spin singlet, whereas otherwise a spin triplet exists as well. The difference between electrons and spinons is, however, that the wavefunction of free electrons factorizes in a spin part, transforming under $SU(2)$, as well as a momentum part, transforming under (lattice) translations; the product of both has to be antisymmetric under permutations. In contrast we cannot factorize spin and momentum of the spinons, as both are incorporated in the representations of $Y(\mathfrak{sl}_2)$ (the lattice translations are implemented as shifts of the spectral parameter ξ). It seems that this entanglement of spin and momentum makes it impossible to implement the fractional statistics by the requirement of a definite transformation law under permutations of the spinons. In fact, this requirement is replaced by the postulate of irreducibility under Yangian transformations.

The spin current of the polarized two-spinon state $|\uparrow\uparrow\rangle$ is easily obtained from Drinfel'd polynomial of the irreducible subrepresentation of $V(\frac{1}{2}, \xi_1) \otimes V(\frac{1}{2}, \xi_2)$, which is given by $P(u) = (u - \xi_1)(u - \xi_2)$. Hence, with (33) and (34) we find

$$H_{1,1} |\uparrow\uparrow\rangle = (\xi_1 + \xi_2 + 1) |\uparrow\uparrow\rangle = (N - 2\mu - 2\nu - 1) |\uparrow\uparrow\rangle, \tag{41}$$

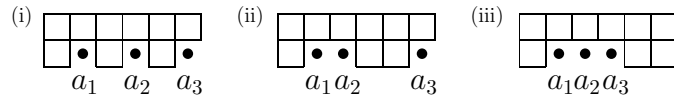
and with (28) we obtain the physical spin current

$$\Lambda^z |\uparrow\uparrow\rangle = \left(\frac{N-2}{2} - \mu - \nu \right) |\uparrow\uparrow\rangle, \tag{42}$$

which equals the result obtained using explicit wavefunctions [18].

6.3. Many-spinon states

If three spinons are present, there are three different cases, which are graphically represented by



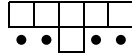
In all three cases we have to determine the irreducible subrepresentation of $V = V(\frac{1}{2}, \xi_1) \otimes V(\frac{1}{2}, \xi_2) \otimes V(\frac{1}{2}, \xi_3)$, where the spectral parameters are given by $\xi_i = a_i - (N + 1)/2$. In the first case V is irreducible and generated by its YHWS $|\uparrow\uparrow\uparrow\rangle$. As \mathfrak{sl}_2 representation we find $\frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$, which is the complete eight-dimensional space $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}$. The $\frac{3}{2}$ is given by the tableau (i) above; the tableaux representing the two $\frac{1}{2}$'s are obtained from (i) by moving either the second or the third spinon to the first row.

In the second case we have $\xi_2 - \xi_1 = 1$ and deduce using section 5.1.ii that the irreducible subrepresentation of $V(\frac{1}{2}, \xi_1) \otimes V(\frac{1}{2}, \xi_2)$ is $V(\mathbf{1}, \xi_1 + \frac{1}{2})$. The remaining tensor product

$V(\mathbf{1}, \xi_1 + \frac{1}{2}) \otimes V(\frac{1}{2}, \xi_3)$ is irreducible, and as sl_2 representation we obtain $\frac{3}{2} \oplus \frac{1}{2}$, which is only six dimensional. The loss of one $\frac{1}{2}$ is reflected by the fact that the second spinon in the tableau (ii) is fixed to the lower row. Note that this result is not affected if a_2 and a_3 were adjacent instead of a_1 and a_2 , although the specific values of the spectral parameters will change.

In the third case the irreducible subrepresentation of $V(\frac{1}{2}, \xi_1) \otimes V(\frac{1}{2}, \xi_2)$ is again given by $V(\mathbf{1}, \xi_1 + \frac{1}{2})$, however, this time the remaining tensor product is reducible as well; and its irreducible subrepresentation is given by $V(\frac{3}{2}, \xi_1 + 1)$. As sl_2 representation we only have $\frac{3}{2}$ which is represented by the tableau (iii).

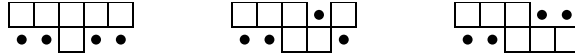
To give a more general example let us first consider a six-site chain and the four-spinon tableau



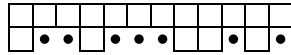
The spin-polarized state in this multiplet, $|\uparrow\uparrow\uparrow\uparrow\rangle$, generates the irreducible subrepresentation of

$$V(\frac{1}{2}, -\frac{5}{2}) \otimes V(\frac{1}{2}, -\frac{3}{2}) \otimes V(\frac{1}{2}, \frac{3}{2}) \otimes V(\frac{1}{2}, \frac{5}{2}), \tag{43}$$

which is given by $V(\mathbf{1}, -2) \otimes V(\mathbf{1}, 2)$. As sl_2 representation this is given by $\mathbf{2} \oplus \mathbf{1} \oplus \mathbf{0}$, which is represented by the tableaux



In the same way we can analyse the space generated by the YHWS of the seven-spinon tableau



where $N = 17$. We couple adjacent spinons according to section 5.1(ii), and find the irreducible subrepresentation to be

$$V(\mathbf{1}, -\frac{11}{2}) \otimes V(\frac{3}{2}, -1) \otimes V(\frac{1}{2}, 5) \otimes V(\frac{1}{2}, 8), \tag{44}$$

which as sl_2 representation reads

$$\mathbf{1} \otimes \frac{3}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{7}{2} \oplus \frac{5}{2} \oplus \frac{5}{2} \oplus \frac{5}{2} \oplus \frac{3}{2} \oplus \frac{3}{2} \oplus \frac{3}{2} \oplus \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}. \tag{45}$$

The corresponding tableaux are easily constructed using that the first, second and fifth spinons are fixed to the lower row, and the fourth spinon can only move to the upper row if the third one does.

The general scheme works as follows. Any spin-polarized m -spinon state is represented by a tableau with all spinons in the second row. The individual spinon momenta are given in terms of the SMN's a_i . The space generated by this YHWS under the action of $Y(sl_2)$ is the irreducible subrepresentation W of the tensor product

$$V = \bigotimes_{i=1}^m V(\frac{1}{2}, \xi_i), \quad \xi_i = a_i - \frac{N+1}{2}, \tag{46}$$

where ξ_i 's have ascending order. In order to construct W one first determines the irreducible subrepresentations of all partial products in (46) which have consecutive spectral parameters $\xi_{i+1} - \xi_i = 1$ using section 5.1(ii). (Note that we can without loss of generality begin with these products as the comultiplication (18) is associative.) The remaining tensor product is then irreducible by repeated application of section 5.1(i) (for a proof see [5]). The sl_2 contents is determined by a straightforward calculation.

To sum up, spinons in the HSM transform under the $Y(\mathfrak{sl}_2)$ representation $V(\frac{1}{2}, \xi)$, where the spectral parameter ξ is via (40) and (8) directly connected to the spinon momentum. All m -spinon states with given individual momenta p_1, \dots, p_m are generated by the YHWS of (46), meaning that they span the irreducible subrepresentation W . The complete Hilbert space is the direct sum of these subspaces. From a mathematical point of view the tableau formalism [28] hence provides an algorithm to determine the \mathfrak{sl}_2 content of the irreducible subrepresentation of a tensor product of fundamental $Y(\mathfrak{sl}_2)$ representations (46) with increasing spectral parameters (the restriction to integer or half-integer spectral parameters ξ_i is no limitation, since all ξ_i 's can be shifted simultaneously and tensor products where the spacings $\xi_j - \xi_i$ are not integers are irreducible [5]).

7. Colorons and representations of $Y(\mathfrak{sl}_3)$

In this section, we will investigate the relation between the $Y(\mathfrak{sl}_3)$ symmetry of the $SU(3)$ HSM and its coloron excitations.

7.1. One-coloron states

Consider a chain with $N = 3M - 1$, $M \in \mathbb{N}$, sites. Then the one-coloron momenta are given by [26]

$$p = \frac{4\pi}{3} - \frac{2\pi}{N} \left(\mu + \frac{1}{3} \right), \quad 0 \leq \mu \leq (N - 2)/3. \tag{47}$$

The $SU(3)$ spin (or colour) currents of a yellow coloron $|y\rangle$, which is a Yangian lowest weight state, are

$$\frac{1}{\sqrt{3}} \Lambda^3 |y\rangle = \Lambda^8 |y\rangle = -\frac{\sqrt{3}}{2} \left(\frac{N - 2}{6} - \mu \right) |y\rangle. \tag{48}$$

In order to apply the representation theory of $Y(\mathfrak{sl}_3)$ it will be appropriate to work with YHWS's, that is magenta colorons $|m\rangle$, instead. As the fundamental representation $V(\bar{\mathbf{3}}, \xi)$ of $Y(\mathfrak{sl}_3)$ can be explicitly realized as evaluation representation (see appendix A), we obtain the spin currents of $|m\rangle$ to be

$$\Lambda^3 |m\rangle = 0, \quad \Lambda^8 |m\rangle = \sqrt{3} \left(\frac{N - 2}{6} - \mu \right) |m\rangle. \tag{49}$$

On the other hand a single coloron is represented by the tableau

			•						
			a						

$$a = N - 3\mu - \frac{1}{2}, \quad 0 \leq \mu \leq \frac{N - 2}{3}.$$

The spectral parameter of $V(\bar{\mathbf{3}}, \xi)$ is determined from the eigenvalue of $H_{2,1}$ when acting on the YHWS $|m\rangle$. Using (31) together with $H_{1,1} |m\rangle = 0$ we find $H_{2,1} |m\rangle = (\sqrt{3}\Lambda^8 - 1/4) |m\rangle = (a - (2N + 3)/4) |m\rangle$. Hence, colorons transform under the $Y(\mathfrak{sl}_3)$ representation $V(\bar{\mathbf{3}}, \xi)$ with spectral parameter

$$\xi = a - \frac{2N + 3}{4}, \quad -\frac{2N - 3}{4} \leq \xi \leq \frac{2N - 5}{4}. \tag{50}$$

Note that although the allowed values for ξ are not symmetrically distributed around zero, the eigenvalues of the physical spin current Λ^8 are.

7.2. Two-coloron states

Compared to the many-spinon states discussed above, the effect of the fractional statistics on many-coloron states is rather complicated. We will discuss in this and the following sections how the requirement of irreducibility under $Y(\mathfrak{sl}_3)$ transformations yields several restrictions on the allowed $SU(3)$ representations for many-coloron states.

Let us first consider two colorons with identical colours like $|mm\rangle = |m\rangle \otimes |m\rangle$. The individual coloron momenta p_1 and p_2 with $p_2 > p_1$ are spaced according to $p_2 - p_1 = 2\pi(2/3 + \ell)/N$, $\ell \in \mathbb{N}_0$ (see (7)). Furthermore, for all pairs of momenta satisfying this condition the $SU(3)$ spin takes the values $\bar{\mathbf{3}} \otimes \bar{\mathbf{3}} = \bar{\mathbf{6}} \oplus \mathbf{3}$, which is graphically reflected by the two tableaux

$$\begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & \bullet & & & \bullet \\ \hline & & & & \\ \hline \end{array} \quad \text{and always} \quad \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \bullet \\ \hline & \bullet & & & \\ \hline & & & & \\ \hline \end{array} \quad a_1 = N - 3\mu - \frac{5}{2}, \quad a_2 = N - 3\nu - \frac{1}{2}, \\
 a_1 < a_2 \quad \quad \quad 0 \leq \nu \leq \mu \leq \frac{N-4}{3}.$$

We note that $a_2 - a_1 \geq 2$ even if the colorons occupy adjacent columns, and that the YHWS $|mm\rangle$ belongs to the left tableau. In order to determine the space generated by $|mm\rangle$ we have to investigate the tensor product $V = V(\bar{\mathbf{3}}, \xi_1) \otimes V(\bar{\mathbf{3}}, \xi_2)$, where $\xi_{1,2} = a_{1,2} - (2N + 3)/4$, respectively. By application of section 5.2(iii), V is irreducible. As \mathfrak{sl}_3 representation we find $V \cong \bar{\mathbf{6}} \oplus \mathbf{3}$, where the $\bar{\mathbf{6}}$ is represented by the left tableau above and the $\mathbf{3}$ by the right tableau. The spin currents of $|mm\rangle$ are obtained from the Drinfel'd polynomials of V , $P_1(u) = 1$ and $P_2(u) = (u - \xi_1)(u - \xi_2)$; they equal the results derived using explicit wavefunctions [26].

There are fundamentally different two-coloron states, namely the ones represented by tableaux like

$$\begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & \bullet & & & \\ \hline & \bullet & & & \\ \hline & & & & \\ \hline \end{array} \quad a_1, a_2 \quad a_1 = N - 3\mu + \frac{1}{2}, \quad a_2 = N - 3\mu - \frac{1}{2}, \\
 0 \leq \mu \leq \frac{N-1}{3}.$$

We have chosen the SMN's to satisfy $a_1 = a_2 + 1$, which is supported by the following consideration: we rewrite the momentum spacing as $\Delta p = p_2 - p_1 = 2\pi(g + \ell)/N$, $\ell \in \mathbb{N}_0$, where g denotes the statistical parameter of the colorons. With the assignment $a_1 = a_2 + 1$ we obtain (we keep the relations $p_i \leftrightarrow a_i$) $g = -1/3$ for the preceding tableau. Moreover, the momentum spacings for the left two-coloron tableau above are also given by $p_2 - p_1 = 2\pi(-1/3 + \ell)/N$ if ℓ takes the values $\ell \geq 1$. Hence, we obtain $g = -1/3$ for all two-coloron states where the $SU(3)$ spins of the colorons are coupled antisymmetrically, i.e., all states represented by the tableaux where the two colorons occupy different rows. The finding $g = 2/3$ for colour-polarized colorons (in general symmetrically coupled) and $g = -1/3$ for colorons with different colours (in general antisymmetrically coupled) is also consistent with what we find by naive state counting. A negative mutual exclusion statistics was also deduced from the dynamical spin susceptibility of the $SU(3)$ HSM calculated by Yamamoto *et al* [35], and observed in conformal field theory spectra analysed by Schoutens [36].

As a consequence, two colorons occupying the same column transform under the $Y(\mathfrak{sl}_3)$ representation $V = V(\bar{\mathbf{3}}, \xi) \otimes V(\bar{\mathbf{3}}, \xi - 1)$ with $\xi = a_1 - (2N + 3)/4$. By section 5.2(iii), V is reducible and the irreducible subrepresentation W does not contain the YHWS of V (which is $|mm\rangle$). As \mathfrak{sl}_3 representation we have $W \cong \mathbf{3}$, i.e., the colorons are coupled antisymmetrically. Hence, if the individual coloron momenta satisfy $|p_2 - p_1| = 2\pi/3N$, we deduce with the

choice $a_1 = a_2 + 1$ and the requirement of irreducibility under $Y(\mathfrak{sl}_3)$ transformations that only the \mathfrak{sl}_3 representation $\mathbf{3}$ exists in the spectrum. This was also found heuristically in the numerical study of the spectrum of the HSM [28], and is consistent with similar results for conformal field theories [27].

Furthermore, it is shown in appendix B that the proper $Y(\mathfrak{sl}_3)$ subrepresentation W of $V(\bar{\mathbf{3}}, \xi) \otimes V(\bar{\mathbf{3}}, \xi - 1)$ is explicitly given by

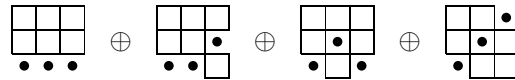
$$W = V(\mathbf{3}, \xi - \frac{1}{2}). \tag{51}$$

This means that W is a highest weight representation with YHWS $|b\rangle \propto |mc\rangle - |cm\rangle$ (see figure 1). We will see below that (51) is necessary and sufficient to build up the complete Hilbert space of the $SU(3)$ HSM with many-coloron states and the restrictions imposed by the fractional statistics through the requirement of irreducibility under $Y(\mathfrak{sl}_3)$ transformations.

At this point we wish to underline that fractional statistics in $SU(n)$ spin chains cannot be implemented by the requirement of a definite transformation law under permutations of the spinons (a one-dimensional representation of the symmetric group), as in specific situations only the antisymmetric spin representations exist in the spinon Hilbert space, whereas in other situations only the symmetric representations exist.

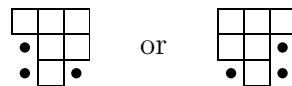
7.3. Three-coloron states

If three colorons are present, there are three different cases to be investigated. In the first case the SMN's $a_{1,2,3}$ satisfy $a_j - a_i \geq 2, i < j$. The tensor product $V(\bar{\mathbf{3}}, \xi_1) \otimes V(\bar{\mathbf{3}}, \xi_2) \otimes V(\bar{\mathbf{3}}, \xi_3)$ is irreducible, hence we find as \mathfrak{sl}_3 representation $\bar{\mathbf{3}} \otimes \bar{\mathbf{3}} \otimes \bar{\mathbf{3}} = \bar{\mathbf{10}} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$, which is graphically represented by the corresponding tableaux:

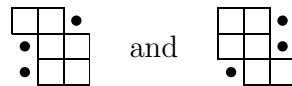


We have drawn the tableaux for the smallest possible system with $N = 6$; the situation is unchanged if the colorons do not occupy adjacent columns. The YHWS $|mmm\rangle$ belongs to the representation $\bar{\mathbf{10}}$ (the left-most tableau).

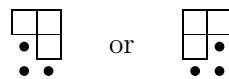
In the second case two of the colorons occupy the same column, whereas the third coloron is separated by at least one column. Graphically we have



The left tableau stands for the \mathfrak{sl}_3 representation containing the YHWS of the tensor product $V(\bar{\mathbf{3}}, \xi_1) \otimes V(\bar{\mathbf{3}}, \xi_1 - 1) \otimes V(\bar{\mathbf{3}}, \xi_2)$, where $\xi_2 - \xi_1 \geq 4$. Using (51) the irreducible subrepresentation of $V(\bar{\mathbf{3}}, \xi_1) \otimes V(\bar{\mathbf{3}}, \xi_1 - 1)$ is $V(\mathbf{3}, \xi_1 - \frac{1}{2})$, hence the remaining tensor product $V(\mathbf{3}, \xi_1 - \frac{1}{2}) \otimes V(\bar{\mathbf{3}}, \xi_2)$ is irreducible by section 5.2(ii). As \mathfrak{sl}_3 representation we find $\mathbf{8} \oplus \mathbf{1}$. The similar result is obtained for the right tableau. The \mathfrak{sl}_3 representations $\mathbf{8}$ are given by the tableaux above; the corresponding singlets are represented by



In the third case all the three colorons are close together, and two of them occupy the same column. Graphically we have



For example, the left tableau is translated into the tensor product $V(\bar{\mathbf{3}}, \xi) \otimes V(\bar{\mathbf{3}}, \xi - 1) \otimes V(\bar{\mathbf{3}}, \xi + 1)$. As in the second case we first construct the irreducible subrepresentation of the first two factors, which is given by $V(\mathbf{3}, \xi - \frac{1}{2})$. The remaining tensor product $V(\mathbf{3}, \xi - \frac{1}{2}) \otimes V(\bar{\mathbf{3}}, \xi + 1)$ is reducible by section 5.2(ii), its irreducible subrepresentation is as \mathfrak{sl}_3 representation given by $\mathbf{8}$. This is reflected in the fact that no singlet tableau with the same SMN's exists. The loss of the singlet in this case was also observed in conformal field theory spectra [27].

7.4. Many-coloron states

Let us first consider four colorons forming two antisymmetrically coupled pairs. Hence we have to investigate the tensor product $V = V(\mathbf{3}, \xi_1) \otimes V(\mathbf{3}, \xi_2)$, where the results of section 5.2(i) apply. If $\xi_2 - \xi_1 > 1$, then V is irreducible and $V \cong \mathbf{6} \oplus \bar{\mathbf{3}}$. If, however, $\xi_2 - \xi_1 = 1$, then V is reducible and its irreducible subrepresentation is, as \mathfrak{sl}_3 representation, given by $\mathbf{6}$. These two situations are graphically represented by the tableaux

$$\begin{aligned} \xi_2 - \xi_1 > 1 : & \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \bullet & \bullet & \bullet \\ \hline \square & \square & \square \\ \hline \end{array} & \text{and} & \begin{array}{|c|c|c|} \hline \square & \square & \bullet \\ \hline \bullet & \bullet & \square \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \\ \xi_2 - \xi_1 = 1 : & \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \square \\ \hline \end{array} & \text{only.} & \end{aligned}$$

If more than four colorons are present, the corresponding product representation of $Y(\mathfrak{sl}_3)$ has to contain more than two fundamental representations. As the representation theory for $Y(\mathfrak{sl}_3)$ is not known in the same detail as that of $Y(\mathfrak{sl}_2)$, we have to restrict ourselves to some illuminating examples. Let us start with the highest-weight tableau

$$\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \quad \text{SMN's : } \begin{aligned} a_1 &= \frac{3}{2}, & a_2 &= \frac{1}{2}, & a_3 &= \frac{5}{2} \\ a_4 &= \frac{9}{2}, & a_5 &= \frac{13}{2}, & a_6 &= \frac{11}{2}. \end{aligned}$$

which stands for the tensor product (we have coupled colorons in the same column already)

$$V(\mathbf{3}, -\frac{11}{4}) \otimes V(\bar{\mathbf{3}}, -\frac{5}{4}) \otimes V(\bar{\mathbf{3}}, \frac{3}{4}) \otimes V(\mathbf{3}, \frac{9}{4}). \tag{52}$$

Using section 5.2.ii the first as well as the third tensor product is reducible; its irreducible subrepresentations are $V(\mathbf{8}, \zeta_1)$ and $V(\mathbf{8}, \zeta_2)$, respectively. We have not determined the spectral parameters explicitly, but expect them to satisfy $-11/4 < \zeta_1 < -5/4$ and $3/4 < \zeta_2 < 9/4$ (in analogy to the $Y(\mathfrak{sl}_2)$ case [5]). Thus we have $\zeta_2 - \zeta_1 > 2$, which causes the irreducibility of $V = V(\mathbf{8}, \zeta_1) \otimes V(\mathbf{8}, \zeta_2)$ [32]. As \mathfrak{sl}_3 representation we find

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{27} \oplus \mathbf{10} \oplus \bar{\mathbf{10}} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}. \tag{53}$$

The irreducibility of V is confirmed by inspection of the allowed tableaux with six colorons and the SMN's given above, which are

$$\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \square \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline \square & \square & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline \square & \bullet & \square \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline \square & \square & \bullet \\ \hline \bullet & \bullet & \square \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline \square & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \square \\ \hline \end{array}$$

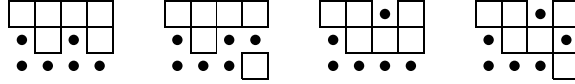
A similar example is obtained from the tableau

$$\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet \\ \hline \end{array} \quad \text{SMN's : } \begin{aligned} a_1 &= \frac{3}{2}, & a_2 &= \frac{1}{2}, & a_3 &= \frac{5}{2} \\ a_4 &= \frac{9}{2}, & a_5 &= \frac{13}{2}, & a_6 &= \frac{11}{2}. \end{aligned}$$

We have to determine the irreducible subrepresentation of the tensor product

$$V(\mathbf{3}, -\frac{11}{4}) \otimes V(\bar{\mathbf{3}}, -\frac{5}{4}) \otimes V(\mathbf{3}, \frac{1}{4}) \otimes V(\bar{\mathbf{3}}, \frac{7}{4}). \tag{54}$$

As before, we obtain as intermediate result $V = V(\mathbf{8}, \zeta_1) \otimes V(\mathbf{8}, \zeta_2)$, but this time the spectral parameters satisfy $-11/4 < \zeta_1 < -5/4$ and $1/4 < \zeta_2 < 7/4$. In particular, they are separated by $1/2$ less than in the preceding example. Inspection of the allowed tableaux with six colorons and the given SMN's, which are



shows that the irreducible subrepresentation of V is, as representation of \mathfrak{sl}_3 , given by $\mathbf{27} \oplus \mathbf{10} \oplus \mathbf{\bar{10}} \oplus \mathbf{8}$. The difference to (53) implies that V is reducible. Physically, the fractional statistics of the colorons restricts the allowed $SU(3)$ representations more than above, as the individual coloron momenta are closer together.

As a final example consider the six-coloron states with SMN's $a_1 = 3/2, a_2 = 7/2, a_3 = 5/2, a_4 = 9/2, a_5 = 7/2$ and $a_6 = 11/2$. For this set of SMN's there exist only the two tableaux



i.e., the irreducible subrepresentation of

$$V(\bar{\mathbf{3}}, -\frac{9}{4}) \otimes V(\mathbf{3}, -\frac{3}{4}) \otimes V(\mathbf{3}, \frac{1}{4}) \otimes V(\bar{\mathbf{3}}, \frac{7}{4}), \tag{55}$$

should be given by $\mathbf{27} \oplus \mathbf{10}$.

The general scheme for $SU(3)$ works as follows. An m -coloron YHWS is represented by a tableau in which the colorons sit at the bottom of the columns. First, we couple the colorons in the same column, i.e., we construct the representations $V(\mathbf{3}, \zeta)$, where ζ is determined using (50) and (51). The remaining colorons transform under $V(\bar{\mathbf{3}}, \xi)$ with ξ given by (50). The space generated under the action of $Y(\mathfrak{sl}_3)$ by the YHWS is the irreducible subrepresentation W of the tensor product

$$V = \bigotimes_{i=1}^{m'} V(x_i, \xi_i), \quad \xi_1 < \xi_2 < \dots < \xi_{m'}, \tag{56}$$

where x_i denotes either $\mathbf{3}$ or $\bar{\mathbf{3}}$, and m' is the number of occupied columns in the tableau (the number of isolated colorons plus the number of coloron pairs). As \mathfrak{sl}_3 representation, W is given by all tableaux with m colorons possessing the corresponding SMN's.

To sum up, colorons transform under the $Y(\mathfrak{sl}_3)$ representation $V(\bar{\mathbf{3}}, \xi)$, where the spectral parameter ξ is directly connected to the individual coloron momentum. The space of m colorons with momenta p_1, \dots, p_m is generated by the YHWS of (56) as explained above. The restrictions on the $SU(3)$ content of this space are due to the fractional statistics of the colorons. From a mathematical point of view the tableau formalism provides an algorithm to derive the \mathfrak{sl}_3 content of the irreducible subrepresentation of a tensor product of fundamental $Y(\mathfrak{sl}_3)$ representations (56) with increasing spectral parameters. As a by-product, this yields an irreducibility criterion for tensor products of the form (56).

8. Conclusion

In conclusion, we have investigated the relation between the spinon excitations of the Haldane–Shastry model and its Yangian symmetry. Each individual spinon transforms under the fundamental representation of the Yangian. The associated spectral parameter is directly proportional to its momentum. We have obtained a generalized Pauli principle which states

that the spinon Hilbert space is built up by the irreducible subrepresentations of tensor products of these fundamental representations. This enabled us to derive several restrictions on the total spin of many-spinon states. Although the fractional statistics of spinons can be implemented using the representation theory of the Yangian only for spinons in the Haldane–Shastry model, we expect the rules governing the allowed values of the total spin of many-spinon states to be valid for interacting spinons in general spin chains as well.

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Appendix A. Realization of $V(\bar{\mathbf{3}}, \xi)$ as evaluation representation

Consider the representation $V(\bar{\mathbf{3}}, \xi)$ with Drinfel'd polynomials $P_1(u) = 1$ and $P_2(u) = u - \xi$, and denote by $|m\rangle$ its YHWS. Then we have

$$H_{1,0}|m\rangle = H_{1,1}|m\rangle = 0, \quad H_{2,0}|m\rangle = |m\rangle, \quad H_{2,1}|m\rangle = \xi|m\rangle, \quad (\text{A.1})$$

and with (31) we deduce

$$\Lambda^8|m\rangle = \frac{1}{\sqrt{3}}\left(\xi + \frac{1}{4}\right)|m\rangle. \quad (\text{A.2})$$

On the other hand we find with (32) that

$$\begin{aligned} \text{ev}_\zeta(\Lambda^8) &= \zeta J^8 + \frac{1}{2\sqrt{3}}(J^3 J^3 - J^8 J^8) + \frac{1}{4\sqrt{3}}(I^+ I^- + I^- I^+) \\ &\quad - \frac{1}{8\sqrt{3}}(U^+ U^- + U^- U^+ + V^+ V^- + V^- V^+), \end{aligned} \quad (\text{A.3})$$

and hence for the action of Λ^8 on $|m\rangle$ in the evaluation representation ϕ_ζ

$$\Lambda^8|m\rangle = \frac{1}{\sqrt{3}}\left(\zeta - \frac{5}{12}\right)|m\rangle. \quad (\text{A.4})$$

Comparison of (A.2) and (A.4) yields $\zeta = \xi + 2/3$.

Appendix B. Irreducible subrepresentation of $V(\bar{\mathbf{3}}, \xi) \otimes V(\bar{\mathbf{3}}, \xi - 1)$

Consider the tensor product $V = V(\bar{\mathbf{3}}, \xi_1) \otimes V(\bar{\mathbf{3}}, \xi_2)$. V contains a proper $Y(\mathfrak{sl}_3)$ subrepresentation W isomorphic to $\mathbf{3}$ as \mathfrak{sl}_3 representation if and only if $\xi_1 - \xi_2 = 1$ [30]. We wish to determine the Drinfel'd polynomials of W . For that we have to evaluate the actions of $H_{1,1}$ and $H_{2,1}$ on the YHWS $|b\rangle = |m\rangle \otimes |c\rangle - |c\rangle \otimes |m\rangle$, where $|b\rangle \in \mathbf{3}$ and $|m\rangle, |c\rangle \in \bar{\mathbf{3}}$.

First, we obtain from (30)

$$\Lambda^3|b\rangle = \frac{1}{2}H_{1,1}|b\rangle + \frac{1}{8}|b\rangle. \quad (\text{B.1})$$

On the other hand we find the action of Λ^3 on V to be

$$\begin{aligned} \Delta(\Lambda^3) &= \Lambda^3 \otimes 1 + 1 \otimes \Lambda^3 - f^{3ab} J^a \otimes J^b \\ &= \Lambda^3 \otimes 1 + 1 \otimes \Lambda^3 + \frac{1}{2}(I^+ \otimes I^- - I^- \otimes I^+) \\ &\quad - \frac{1}{4}(U^+ \otimes U^- - U^- \otimes U^+ - V^+ \otimes V^- + V^- \otimes V^+). \end{aligned} \quad (\text{B.2})$$

On each factor of V the action of Λ^3 is by (32)

$$\text{ev}_{\xi_{1,2}}(\Lambda^3) = \xi_{1,2} J^3 + \frac{1}{\sqrt{3}} J^3 J^8 - \frac{1}{8}(U^+ U^- + U^- U^+ - V^+ V^- - V^- V^+), \quad (\text{B.3})$$

especially we get

$$\Lambda^3 |m\rangle = 0, \quad \Lambda^3 |c\rangle = \left(\frac{\xi_{1,2}}{2} + \frac{1}{8} \right) |c\rangle. \quad (\text{B.4})$$

Hence, we find

$$\Delta(\Lambda^3) |b\rangle = \left(\frac{\xi_2}{2} + \frac{3}{8} \right) |m\rangle \otimes |c\rangle - \left(\frac{\xi_1}{2} - \frac{1}{8} \right) |c\rangle \otimes |m\rangle = \left(\frac{\xi_1}{2} - \frac{1}{8} \right) |b\rangle. \quad (\text{B.5})$$

The last equality is valid if and only if $\xi_1 - \xi_2 = 1$, i.e., when the $Y(\mathfrak{sl}_3)$ subrepresentation W is indeed isomorphic to $\mathbf{3}$. With (B.1) we deduce using $\xi \equiv \xi_1 = \xi_2 + 1$ that $H_{1,1} |b\rangle = (\xi - 1/2) |b\rangle$. As W can be constructed as evaluation representation using (32) we obtain $\Lambda^8 = \Lambda^3/\sqrt{3}$, and with (31) as well as (B.5) we find $H_{2,1} |b\rangle = 0$. Hence, the Drinfel'd polynomials of W are $P_1(u) = u - (\xi - 1/2)$ and $P_2(u) = 1$.

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